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Spaces of rational maps and the Stone–Weierstrass theorem

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To Alberto Verjovsky on the occasion of his 60th birthday

Abstract

It is shown that Segal's theorem on the spaces of rational maps from \mathbf{CP}^1 to \mathbf{CP}^n can be extended to the spaces of continuous rational maps from \mathbf{CP}^m to \mathbf{CP}^n for any $m \leq n$. The tools are the Stone–Weierstrass theorem and Vassiliev's machinery of simplicial resolutions.

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1. Introduction

Let X and Y be two topological spaces with some additional structure. Maps from X to Y that preserve the structure form a subspace in the space of all continuous maps from X to Y . One may ask whether this subspace gives some kind of a “topological approximation” to the space of all continuous maps.

For example, if X is a Stein manifold and Y is a complex manifold with a dominating spray (homogeneous spaces and complements in \mathbf{C}^N of algebraic subvarieties of codimension at least two satisfy this condition), the inclusion of the space of holomorphic maps from X to Y into the space of all continuous maps from X to Y is a weak homotopy equivalence. This statement is a corollary of the Oka–Grauert principle proved by Gromov [3].

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Another situation which has been extensively studied is the case when X and Y are compact complex manifolds and X has dimension one. Segal in [11] proved, among other things, that the inclusion of the space of (based) rational maps of degree d from \mathbf{CP}^1 to \mathbf{CP}^n into the two-fold loop space of \mathbf{CP}^n is a homotopy equivalence up to dimension $(2n - 1)d$. The results of [11] were later extended by various authors: see, for example [1,4,5,7,9]. In all these generalizations X is a curve. Apparently, the only published works that address the case $\dim X > 1$ are the paper [6] which deals with the space of holomorphic maps from $\mathbf{CP}^1 \times \mathbf{CP}^1$ to the Grassmannian of two-dimensional planes in \mathbf{C}^N , and the work [8] describing spaces of linear maps between complex projective spaces.

Segal conjectured in [11] that the stability theorem that he proved for spaces of rational curves can be extended to the spaces of maps from \mathbf{CP}^m to \mathbf{CP}^n for $m > 1$. Such a generalization will be discussed in this paper.

A continuous rational map from \mathbf{CP}^m to \mathbf{CP}^n can be given by a collection of $n + 1$ complex homogeneous polynomials of the same degree and with no common zero. Up to a constant, such a representation of a map by polynomials is unique. If $m > n$, all continuous rational maps are constant, so we will always assume that $m \leq n$. By the degree of a rational map we will mean the degree of the polynomials that define it. More generally, a continuous map between complex projective spaces induces multiplication by an integer in the second cohomology. We will call this integer the degree of the corresponding map.¹ In what follows, all rational maps will be continuous.

Let $\mathbf{Rat}_f^{m,n}$ be the set of all rational maps of degree d from \mathbf{CP}^m to \mathbf{CP}^n that restrict to a given map

$$f : \mathbf{CP}^{m-1} \rightarrow \mathbf{CP}^n$$

on a fixed hyperplane $\mathbf{CP}^{m-1} \subset \mathbf{CP}^m$. Fixing the coefficients of the polynomials that define f one obtains a bijection between $\mathbf{Rat}_f^{m,n}$ and a subset of a complex affine space: a rational map under this bijection is sent to the set of the coefficients of the polynomials defining it. Thus $\mathbf{Rat}_f^{m,n}$ acquires a topology from the complex affine space.

Consider the space of all continuous maps from \mathbf{CP}^m to \mathbf{CP}^n that restrict to a given map f on \mathbf{CP}^{m-1} . This space is homotopy equivalent to the $2m$ -fold loop space of \mathbf{CP}^n and in what follows will be denoted by $\Omega^{2m}\mathbf{CP}^n$.

Theorem 1. *The map*

$$\mathbf{Rat}_f^{m,n} \rightarrow \Omega^{2m}\mathbf{CP}^n,$$

given by the inclusion of rational maps into the space of all continuous maps from \mathbf{CP}^m to \mathbf{CP}^n that restrict to a given map f of degree d on a fixed hyperplane, induces an isomorphism in homology in all dimensions smaller than

$$(2n - 2m + 1) \left(\left\lfloor \frac{d + 1}{2} \right\rfloor + 1 \right).$$

If $m < n$ it also induces isomorphisms of homotopy groups in these dimensions. Here $[x]$ denotes the integer part of x .

¹ This corresponds to the algebraic rather than topological degree.

Theorem 1 implies a similar result for the spaces of “free” maps. Denote by $\text{Map}_d(m, n)$ the space of all continuous maps from \mathbf{CP}^m to \mathbf{CP}^n of degree d , and let $\mathbf{Rat}_{(d)}^{m,n}$ be its subspace formed by all rational maps.

Theorem 2. *In the statement of Theorem 1 the spaces $\mathbf{Rat}_f^{m,n}$ and $\Omega^{2m}\mathbf{CP}^n$ can be replaced by $\mathbf{Rat}_{(d)}^{m,n}$ and $\text{Map}_d(m, n)$, respectively.*

The proof of Theorem 1 consists of constructing a sequence of topological spaces and “stabilization” maps between them which starts with $\mathbf{Rat}_f^{m,n}$ and has the loop space $\Omega^{2m}\mathbf{CP}^n$ as its colimit. We shall then see that all stabilization maps in this sequence induce isomorphisms in homology up to a certain dimension. This, together with the fact that for $m < n$ the spaces $\mathbf{Rat}_f^{m,n}$ are simply-connected, will imply Theorem 1.

In the next section the stabilization maps are defined and it is shown that the “stabilized” space of rational maps is homotopy equivalent to $\Omega^{2m}\mathbf{CP}^n$. The trick behind our construction of the stabilization maps is to place the problem in a real rather than complex context and use the Stone–Weierstrass Theorem. Section 3 is a brief review of simplicial resolutions. These are then used in Section 4 to construct and describe the Vassiliev spectral sequences converging to the homology of the spaces $\mathbf{Rat}_f^{m,n}$. We follow Vassiliev’s methods as described in [12,13], the only minor novelty being our use of degenerate simplicial resolutions. The case of the free maps is treated in Section 5. Some comments are given in the last section.

2. Stabilization maps and the Stone–Weierstrass theorem

There are two ways of defining stabilization maps for the spaces of rational maps. One of these constructions generalizes Segal’s stabilization maps; we are not going to discuss it here. The stabilization we shall use can be described as “real stabilization”. In order to define it, we need to consider a wider class of maps between projective spaces.

Define a (p, q) -polynomial to be a homogeneous polynomial in “holomorphic” and “anti-holomorphic” variables which is of degree p in the holomorphic and of degree q in the anti-holomorphic variables. By a (p, q) -map we will mean a map from \mathbf{CP}^m to \mathbf{CP}^n given by a collection of $n + 1$ (p, q) -polynomials with no common zero. Here, the holomorphic variables are the homogeneous coordinates z_i in \mathbf{CP}^m and the anti-holomorphic variables are their conjugates. In this terminology $(d, 0)$ -maps are precisely the rational maps of degree d . In general, two representations of a (p, q) -map by collections of (p, q) -polynomials need not coincide up to a constant but rather up to multiplication by a positive function. Thus a (p, q) -map can be also thought of as a $(p + 1, q + 1)$ -map: just multiply each of the polynomials defining the map by $z_0\bar{z}_0 + \cdots + z_m\bar{z}_m$. The degree of a (p, q) -map is readily seen to be equal to $p - q$.

Let $\mathbf{Rat}_f(p, q)$ be the space of (p, q) -maps from \mathbf{CP}^m to \mathbf{CP}^n that restrict to a given map f of degree $p - q$ on a hyperplane $\mathbf{CP}^{m-1} \subset \mathbf{CP}^m$. In what follows we shall always assume that this hyperplane is given by the equation $z_m = 0$. The topology on $\mathbf{Rat}_f(p, q)$ is the topology of a subset of the space of all continuous maps. Clearly, $\mathbf{Rat}_f(\deg f, 0) = \mathbf{Rat}_f^{m,n}$. Considering (p, q) -maps as $(p + 1, q + 1)$ -maps one obtains the inclusion

$$\mathbf{Rat}_f(p, q) \hookrightarrow \mathbf{Rat}_f(p + 1, q + 1).$$

We define $\mathbf{Rat}_f(d + \infty, \infty)$ to be the union of the spaces $\mathbf{Rat}_f(d + k, k)$ for all $k \geq 0$.

The space of all continuous maps from \mathbf{CP}^m to \mathbf{CP}^n that restrict to f on a fixed hyperplane is homotopy equivalent to the space $\Omega^{2m}\mathbf{CP}^n$. Indeed, \mathbf{CP}^m can be obtained from a $2m$ -dimensional disk D^{2m} by making identifications on the boundary of D^{2m} ; namely, by collapsing the fibres of the Hopf map $\partial D^{2m} \rightarrow \mathbf{CP}^{m-1}$. Hence, a map from \mathbf{CP}^m to \mathbf{CP}^n can be thought of as defined on D^{2m} ; the space of all maps from $D^{2m} \rightarrow \mathbf{CP}^n$ which restrict to the same map on ∂D^{2m} is readily seen to be homotopy equivalent to $\Omega^{2m}\mathbf{CP}^n$.

Proposition 3. *The natural inclusion of $\mathbf{Rat}_f(d + \infty, \infty)$ into the space $\Omega^{2m}\mathbf{CP}^m$ of all continuous maps from \mathbf{CP}^m to \mathbf{CP}^n that restrict to f on a fixed hyperplane, is a homotopy equivalence.*

Proposition 3 is a consequence of the following statement:

Lemma 4. *Let X be a finite CW-complex. Any continuous map*

$$F : \mathbf{CP}^m \times X \rightarrow \mathbf{CP}^n$$

can be uniformly approximated with respect to the Fubini-Study metric on \mathbf{CP}^n by maps whose restriction to $\mathbf{CP}^m \times \{x\}$ for any $x \in X$, is a (p, q) -map for some p, q . Moreover, if the restriction of F to $\mathbf{CP}^{m-1} \times \{x\}$ is a (p, q) -map for all $x \in X$, the approximating maps can be chosen so as to coincide with F on $\mathbf{CP}^{m-1} \times X$.

Proof of Proposition 3. For any compact Riemannian manifold M and any space Y there exists $\varepsilon > 0$ such that any two maps $Y \rightarrow M$ that are uniformly ε -close, are homotopic. Thus, Lemma 4 implies that the map of the homotopy groups

$$\pi_k \mathbf{Rat}_f(d + \infty, \infty) \rightarrow \pi_k \Omega^{2m}\mathbf{CP}^n$$

is surjective for all k . Indeed, setting $X = S^k$ we get that any element of $\pi_k \Omega^{2m}\mathbf{CP}^n$ can be approximated by a homotopy class in the space of (p, q) -maps. On the other hand, this homomorphism is also injective, as any homotopy that goes through continuous maps can be approximated by a homotopy through (p, q) -maps (set $X = S^k \times [0, 1]$). Both spaces of maps have homotopy types of CW-complexes, so by the Whitehead theorem they are homotopy equivalent. \square

Lemma 4 is a corollary of the Stone–Weierstrass Theorem for vector bundles:

Theorem 5. *Let E be a locally trivial real vector bundle over a compact space Y , $s_\alpha : Y \rightarrow E$ —a set of its sections, and let A be a subalgebra of the \mathbf{R} -algebra $C(Y)$ of continuous real-valued functions on Y . Suppose that*

- *the subalgebra A separates points of Y , that is, for any pair $x, y \in Y$ there exists $h \in A$ such that $h(x) \neq h(y)$;*
- *for any $y \in Y$ there exists $h \in A$ such that $h(y) \neq 0$;*
- *for any $y \in Y$ the fibre of E over x is spanned by $s_\alpha(y)$.*

Then the A -module generated by the s_α is dense in the space of all continuous sections of E .

The above version of the Stone–Weierstrass Theorem follows from the usual² Stone–Weierstrass Theorem and the existence of a partition of unity subordinate to the open cover of Y trivialising E .

Proof of Lemma 4. Any continuous map $F : \mathbf{CP}^m \times X \rightarrow \mathbf{CP}^n$ can be given by a collection of $n + 1$ sections of some line bundle E_F over $\mathbf{CP}^m \times X$. Namely, E_F is the pullback of the tautological line bundle on \mathbf{CP}^n with respect to F .

Choose an open cover of X by contractible sets U_β and a partition of unity ρ_β subordinate to U_β , with β belonging to some finite index set. Let d be the degree of the restriction of F to $\mathbf{CP}^m \times \{x\}$ for $x \in X$. The restriction of E_F to each subspace of the form $\mathbf{CP}^m \times U_\beta$ is isomorphic to the pullback of the d th power of the tautological line bundle on \mathbf{CP}^m ; we shall assume that an explicit identification of these two line bundles is chosen for each β .

For each β consider the $2m + 2$ sections of E_F given over $\mathbf{CP}^m \times U_\beta$ by $\rho_\beta z_i^d$ and $\rho_\beta \sqrt{-1} \bar{z}_i^d$ and extended by zero to all of $\mathbf{CP}^m \times X$; here $0 \leq i \leq m$. Denote by σ the set of all such sections.

Let A_0 be the algebra of functions on \mathbf{CP}^m generated by

$$\frac{z_i \bar{z}_i}{z_0 \bar{z}_0 + \cdots + z_m \bar{z}_m},$$

for $0 \leq i \leq m$. The first part of Lemma 4 is then recovered from Theorem 5 applied to E_F : the set of sections s_α is taken to coincide with σ and A is taken to be the algebra of all continuous functions on $\mathbf{CP}^m \times X$ whose restriction to $\mathbf{CP}^m \times \{x\}$ for any $x \in X$ belongs to A_0 .

In order to verify the second part of the lemma we shall show that a sufficiently good approximation to F can be modified so as to coincide with F on $\mathbf{CP}^{m-1} \times X$.

Let S and P be two sections of E_F whose restrictions to $\mathbf{CP}^{m-1} \times \{x\}$ and $\mathbf{CP}^m \times \{x\}$, respectively, are (p, q) -polynomials for all $x \in X$. Denote by s and p the restrictions of S and P to $\mathbf{CP}^{m-1} \times X$. One can think of s and p as defined on all $\mathbf{CP}^m \times X$ and being independent of z_m and \bar{z}_m .

Consider a family P_t of sections of E_F with $t \in [0, 1]$:

$$P_t = P + t(s - p). \quad \square$$

For any t the restriction of P_t to any $\mathbf{CP}^m \times \{x\}$ is a (p, q) -polynomial. Clearly, $P_0 = P$ and the restriction of P_1 to any $\mathbf{CP}^{m-1} \times \{x\}$ coincides with that of S . Moreover, if $|P - S| < \varepsilon$ then $|p - s| < \varepsilon$ and, hence, $|P_t - S| < 2\varepsilon$ in the standard metric that the fibres of E_F inherit from the tautological line bundle on \mathbf{CP}^n .

Now, let F_j , $0 \leq j \leq n$, be a set of sections of E_F that define the map F . Choose $\varepsilon > 0$ and find δ such that any map G from $\mathbf{CP}^m \times X$ to \mathbf{CP}^n given by a set of sections G_j of E_F with $|F_j - G_j| < \delta$, is uniformly ε -close to F . Suppose that we have found $\delta/2$ -approximations F'_j by families of (p, q) -polynomials for each of the F_j . Applying the above construction with $P = F'_j$ and $S = F_j$ we obtain an ε -approximation to F which coincides with F on $\mathbf{CP}^{m-1} \times X$ and whose restriction to any $\mathbf{CP}^m \times \{x\}$ is a (p, q) -map.

² see [10].

3. Simplicial resolutions

Let $h : X \rightarrow Y$ be a finite-to-one surjective map of topological spaces and let i be an embedding of X into \mathbf{R}^N for some N . A *simplicial resolution* associated to the map h with respect to the embedding i , is a subspace X^Δ of $\mathbf{R}^N \times Y$ together with the projection map $h^\Delta : X^\Delta \rightarrow Y$. The points of X^Δ are pairs (t, y) with $y \in Y$ and t belonging to the convex hull of the set $i \circ h^{-1}(y)$ in \mathbf{R}^N . The space X is a subspace of X^Δ ; the restriction of h^Δ to X coincides with the original map f . We say that a simplicial resolution is *non-degenerate* if for each $y \in Y$ any k points of the set $i \circ h^{-1}(y)$ span a $(k - 1)$ -dimensional affine subspace of \mathbf{R}^N . Sometimes we will use the term “simplicial resolution” for the space X^Δ , which should not lead to confusion.

The fibres of the projection map h^Δ are contractible, being convex polyhedra. We will need simplicial resolutions in the situation when X and Y are closed semialgebraic subsets of \mathbf{R}^N and h and i are polynomial maps; from now on we shall assume that this is the case. Under these circumstances h^Δ will always be a homotopy equivalence; hence, the problem of computing the homotopy type of Y is equivalent to the same problem for X^Δ .

It is clear that any two non-degenerate simplicial resolutions associated to the same map but with respect to different embeddings of X into \mathbf{R}^N are homeomorphic over Y . This statement is generally false without the non-degeneracy assumption.

There is an increasing filtration

$$X_1 \subset X_2 \subset \dots \subset X^\Delta$$

on any simplicial resolution associated to a map $h : X \rightarrow Y$. Assume first that $h^\Delta : X^\Delta \rightarrow Y$ is non-degenerate. Then for any $y \in Y$ its inverse image $(h^\Delta)^{-1}(y)$ is a simplex; the subspace $X_k \subset X^\Delta$ is then defined as the union of the $(k - 1)$ -skeleta of these simplices over all $y \in Y$. In particular, $X_1 = X$.

Now let $\tilde{X}^\Delta, X^\Delta \rightarrow Y$ be two simplicial resolutions associated to the same map $h : X \rightarrow Y$ and suppose that $\tilde{X}^\Delta \rightarrow Y$ is non-degenerate. There exists the unique map

$$\pi : \tilde{X}^\Delta \rightarrow X^\Delta$$

over Y which commutes with the inclusions of X into \tilde{X}^Δ and X^Δ and which is affine over all points of Y . The increasing filtration on X^Δ is defined as the image of the increasing filtration on \tilde{X}^Δ under π . This definition does not depend on the choice of a non-degenerate resolution. Indeed, if both $\tilde{X}^\Delta, X^\Delta \rightarrow Y$ are non-degenerate, the canonical map from \tilde{X}^Δ to X^Δ not only is a homeomorphism, but also respects the increasing filtration.

The same construction can be carried out if the map $h : X \rightarrow Y$ is not finite-to-one. However, in this situation every simplicial resolution, as defined above, is necessarily degenerate. We define a non-degenerate resolution as follows:

Let $i_k : X \rightarrow \mathbf{R}^{N_k}$ be an embedding such that for any $2k$ distinct points in X their images in \mathbf{R}^{N_k} span an affine subspace of dimension $2k - 1$. Then for all $j \leq k$, for any $y \in Y$ and for any j points in the set $i_k \circ h^{-1}(y)$ their convex hull is an $(j - 1)$ -simplex; moreover, the $(j - 1)$ -simplices corresponding to disjoint sets of points in $i_k \circ h^{-1}(y)$ are disjoint. Now, define X_k to be the union of all the convex hulls in \mathbf{R}^{N_k} of the subsets of cardinality at most k of $i_k \circ h^{-1}(y)$ for all $y \in Y$. There exists a natural extension of h to X_k .

For all $i \leq j$ the space X_i can be considered as a subspace of X_j . The non-degenerate simplicial resolution $(\tilde{X}^A, \tilde{h}^A)$ associated to h is then defined as the union of all X_i together with the extension of h to $\cup X_i$; the spaces X_i form the increasing filtration on \tilde{X}^A .

As before, the non-degenerate simplicial resolution does not depend on the embeddings i_k and it is universal in the sense that for any other simplicial resolution X^A there is a unique map $\tilde{X}^A \rightarrow X^A$ over Y which is identity on $X \in \tilde{X}^A$, X^A and which is affine on the fibres. Thus, we have an increasing filtration on any simplicial resolution of h even in the case when h is not finite-to-one.

Finally, let us state one obvious but important property of non-degenerate simplicial resolutions. Assume there is a commutative square

$$\begin{array}{ccc} X & \rightarrow & X' \\ \downarrow h & & \downarrow h' \\ Y & \rightarrow & Y' \end{array}$$

with h and h' surjective. Then there is an induced filtration-preserving map of non-degenerate simplicial resolutions associated to h and h' .

4. The Vassiliev spectral sequence

The space $\mathbf{Rat}_f(p, q)$ is defined as a subspace of $\Omega^{2m} \mathbf{CP}^n$. In order to apply the machinery of simplicial resolutions we shall replace it with a homotopy equivalent space $\overline{\mathbf{Rat}}_f(p, q)$ defined as a complement to an algebraic subvariety of an affine space.

Let us fix the coefficients of the (p, q) -polynomials f_i defining the (p, q) -map $f : \mathbf{CP}^{m-1} \rightarrow \mathbf{CP}^n$. Denote by

- $W_{p,q}^i$ —the complex affine space of all (p, q) -polynomials that restrict to f_i on the hyperplane $z_m = 0$;
- $W_{p,q}$ —the Cartesian product of the $W_{p,q}^i$ for $0 \leq i \leq n$;
- $N_{p,q}$ —the complex dimension of $W_{p,q}$.

Define $\overline{\mathbf{Rat}}_f(p, q) \subset W_{p,q}$ to be the space of all $(n+1)$ -tuples of (p, q) -polynomials which have no common zero. The natural map

$$\overline{\mathbf{Rat}}_f(p, q) \rightarrow \mathbf{Rat}_f(p, q)$$

is not a homeomorphism. However, it is obviously onto and its fibres are convex and, hence, contractible. It is easy to see that the above map is, in fact, a homotopy equivalence. The space $\overline{\mathbf{Rat}}_f(p, q)$ is the complement of a discriminant Σ in $W_{p,q}$, which consists of $(n+1)$ -tuples of (p, q) -polynomials that all have a common zero. It has complex codimension $n - m + 1$ in $W_{p,q}$ so the spaces $\mathbf{Rat}_f^{m,n}$ are simply-connected if $m < n$.

In what follows the notation \mathbf{C}^m will be used for the affine chart $z_m = 1$ in \mathbf{CP}^m . Let V be the complex vector space of dimension $\binom{p+m}{m} \binom{q+m}{m}$ spanned by all monomials in z_i and \bar{z}_i (with $0 \leq i < m$) of degree at most p in z_i and at most q in \bar{z}_i . We shall denote by $v_{p,q}$ the Veronese-type embedding of \mathbf{C}^m into V which sends a point $x = (z_0, \dots, z_{m-1})$ to the point whose coordinates are the values of the corresponding monomials at x .

Let $Z \subset W_{p,q} \times \mathbf{C}^m$ be the set

$$(F_0, F_1, \dots, F_n, x \mid F_i(x) = 0 \text{ for all } i).$$

There is a projection map $Z \rightarrow \Sigma$ which “forgets” the point x where all the polynomials F_i vanish. Denote by Z^A the space of the simplicial resolution associated to this map, with respect to the embedding $Z \hookrightarrow W_{p,q} \times V$ which sends (F_i, x) to $(F_i, v_{p,q}(x))$. We shall also use the corresponding non-degenerate resolution, which will be denoted by \tilde{Z}^A . The resolution Z^A will be, however, of greater importance to us as it satisfies the inequality (2) below.

Let \widehat{Z} , \widehat{Z}^A and $\widehat{\Sigma}$ be the one-point compactifications of Z , Z^A and Σ , respectively. The projection of Z^A onto Σ extends to a homotopy equivalence between \widehat{Z}^A and $\widehat{\Sigma}$. There is an increasing filtration

$$\widehat{Z}_0 \subset \widehat{Z}_1 \subset \dots \subset \widehat{Z}^A$$

coming from the filtration Z_r on Z^A : the term \widehat{Z}_0 is the added point and \widehat{Z}_r is equal to $Z_r \cup \widehat{Z}_0$. In particular, for all $r > 0$ the spaces $\widehat{Z}_r \setminus \widehat{Z}_{r-1}$ and $Z_r \setminus Z_{r-1}$ coincide.

The filtration on \widehat{Z}^A gives rise to a spectral sequence converging to the cohomology of Σ :

$$E_1^{r,s} = H^{r+s}(\widehat{Z}_r, \widehat{Z}_{r-1}, \mathbf{Z}),$$

where \widehat{Z}_{-1} is the empty set. The cohomology of $\widehat{\Sigma}$ is related by the Alexander duality to the homology of $\mathbf{Rat}_f(p, q)$:

$$H^r(\widehat{\Sigma}, \mathbf{Z}) \simeq H_{2N_{p,q}-r-1}(\overline{\mathbf{Rat}}_f(p, q))$$

so the following spectral sequence converges to the homology of $\overline{\mathbf{Rat}}_f(p, q)$:

$$E_{-r,s}^1 = H^{2N_{p,q}+r-s-1}(\widehat{Z}_r, \widehat{Z}_{r-1}, \mathbf{Z}) \quad (1)$$

with $r, s \geq 0$.

For $r > 0$ the space $\widehat{Z}_r / \widehat{Z}_{r-1} = Z_r / Z_{r-1}$ is the one-point compactification of the space $Z_r \setminus Z_{r-1}$, which admits a rather explicit description, at least for small values of r .

A point of Z_r is given by specifying (a) a point in $W_{p,q}$ which corresponds to $n+1$ polynomials F_i vanishing simultaneously on a non-empty subset of \mathbf{C}^m ; (b) a point in V which belongs to the convex hull of at most r points of the form $v_{p,q}(x_j)$ where x_j are distinct points of \mathbf{C}^m on which all the F_i vanish.

The condition that a polynomial in $W_{p,q}^i$ vanishes at a given point gives one linear inhomogeneous condition on its coefficients. More generally, the condition that a (p, q) -polynomial in $W_{p,q}^i$ vanishes at r distinct points x_j produces exactly r independent conditions on its coefficients if and only if the convex hull of the points $v_{p,q}(x_j)$ in V is an $(r-1)$ -dimensional simplex.

Hence,

$$\begin{aligned} \dim Z_r \setminus Z_{r-1} &\leq 2(N_{p,q} - r(n+1)) + 2mr + r - 1 \\ &= 2N_{p,q} - 2r(n-m+1) + r - 1. \end{aligned} \quad (2)$$

In general, little else can be said without a sophisticated analysis. For example, a set of distinct points in \mathbf{C}^m can give rise to a set of linearly dependent conditions on the coefficients of a polynomial vanishing at these points. However, the following is true:

Proposition 6. For all $r \leq [(p+1)/2]$ the space $Z_r \setminus Z_{r-1}$ is homeomorphic to a real vector bundle of rank

$$2N_{p,q} - 2(n+1)r + r - 1$$

over the configuration space $C_r(\mathbb{C}^m)$ of r distinct unordered points in \mathbb{C}^m .

The proof of this statement is based on the following fact:

Lemma 7. If $r \leq p+1$ the images in V under $v_{p,q}$ of any set of r distinct points in \mathbb{C}^m span an $(r-1)$ -simplex.

Indeed, by a linear change of coordinates in \mathbb{C}^m one can achieve that the values of the coordinate z_0 for all r points are pairwise distinct; then the Vandermonde matrix constructed of the powers of z_0 is non-degenerate.

Remark. Lemma 7 implies that for $r \leq p+1$ the condition that a polynomial $F_i \in W_{p,q}^i$ vanishes at r given points of \mathbb{C}^m determines r independent linear inhomogeneous conditions on the coefficients of F_i . In general, these conditions may be incompatible. The case $m=1$, $q=0$ is an example: a non-trivial polynomial of degree p in one variable cannot have $p+1$ roots. However, for $r \leq p$ the r affine hyperplanes in $W_{p,q}^i$ defined by the vanishing conditions at r points are necessarily in general position. This is proved by the same argument as Lemma 7.

Proof of Proposition 6. It follows from Lemma 7 that if $r \leq [(p+1)/2]$ then any two $(r-1)$ -simplices in V whose vertices are in the image of $v_{p,q}$ are either disjoint or have a common face. Hence, for these values of r , given a point in the interior of the convex hull of r distinct points of the form $v_{p,q}(x_j)$ in V one can determine the points x_j up to order. Therefore, there exists a map

$$Z_r \setminus Z_{r-1} \rightarrow C_r(\mathbb{C}^m)$$

which keeps track of r -tuples of points in \mathbb{C}^m on which the polynomials F_i vanish.

It was mentioned before that the condition for a polynomial to vanish at a given point, determines an affine hyperplane in $W_{p,q}^i$. By the remark to Lemma 7, any set of r points in \mathbb{C}^m with $r \leq p$, determines a set of hyperplanes in general position, hence their intersection has complex codimension r in $W_{p,q}^i$. Therefore, for all $r \leq [(p+1)/2]$ the fibre of the projection map of $Z_r \setminus Z_{r-1}$ to $C_r(\mathbb{C}^m)$ is a product of a complex vector space of complex dimension $N_{p,q} - (n+1)r$, with an interior of an $(r-1)$ -simplex. Verifying the local triviality property is straightforward. \square

The cohomology groups of the space Z_r/Z_{r-1} are the same as those of a reduced Thom space of a vector bundle of rank $2N_{p,q} - 2(n+1)r + r - 1$ over the one-point compactification $\widehat{C}_r(\mathbb{C}^m)$ of $C_r(\mathbb{C}^m)$. In particular,

$$H^i(\widehat{C}_{r+1}(\mathbb{C}^m), \mathbf{Z}) = \widetilde{H}^{2N_{p,q} - 2(n+1)r + r - 1 + i}(Z_r/Z_{r-1}, \mathbf{Z})$$

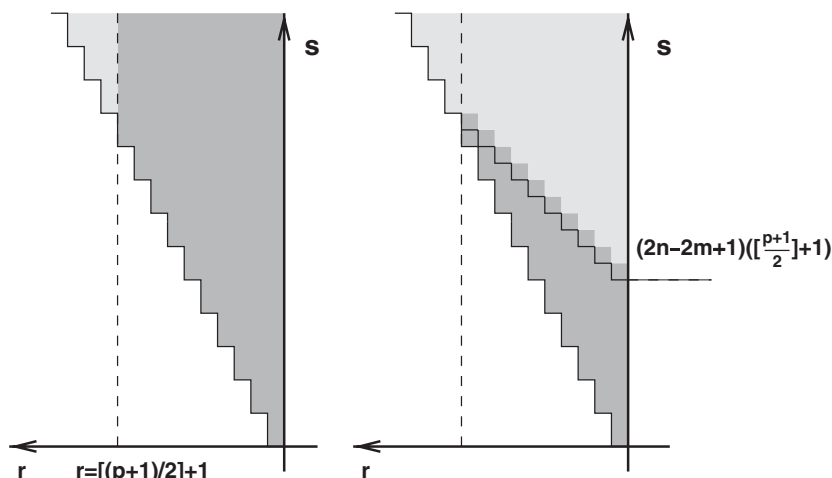


Fig. 1. The terms E^1 and E^∞ of the spectral sequence converging to the homology of $\mathbf{Rat}_f(p, q)$. The group $E_{a,b}^*$ is placed in the square $a-1 < x < a$, $b < y < b+1$. The shaded entries are stable.

for all i , and, hence,

$$E_{-r,s}^1 = H^{2(n+1)r-s}(\widehat{C}_r(\mathbf{C}^m), \mathbf{Z})$$

for all $0 < r \leq [(p+1)/2]$ and all s . Note that this expression does not depend on p and q .

The term $E_{-r,s}^1$ of the spectral sequence for the homology of $\mathbf{Rat}_f(p, q)$ is shown in the Fig. 1. It follows from (1) and (2) that all non-zero entries are situated in the sector $-r \leq 0, s \geq 2(n-m+1)r$. The entries in the strip $r \leq [(p+1)/2]$ are “stable” in the sense that they do not depend on p and q and are preserved by the stabilization maps. Let us make the last statement more precise.

Let us introduce the notation $\Sigma_{p,q}$ and $Z_{p,q}$ instead of Σ and Z , respectively. The inclusion map

$$\mathbf{Rat}_f(p, q) \rightarrow \mathbf{Rat}_f(p+1, q+1)$$

can be lifted to a map

$$\overline{\mathbf{Rat}}_f(p, q) \rightarrow \overline{\mathbf{Rat}}_f(p+1, q+1)$$

which multiplies all the $(n+1)(p, q)$ -polynomials by $z_0\bar{z}_0 + \dots + z_m\bar{z}_m$. This map, in fact, extends to a map $W_{p,q} \rightarrow W_{p+1,q+1}$; it sends the discriminant $\Sigma_{p,q}$ to $\Sigma_{p+1,q+1}$. There is also a corresponding map from $Z_{p,q}$ to $Z_{p+1,q+1}$. However, there is no induced map between the degenerate simplicial resolutions $Z_{p,q}^\Delta$.

Consider the non-degenerate simplicial resolution $\tilde{Z}^\Delta = \tilde{Z}_{p,q}^\Delta$. There are maps

$$Z_{p,q}^\Delta \xleftarrow{\pi_1} \tilde{Z}_{p,q}^\Delta \xrightarrow{\zeta} \tilde{Z}_{p+1,q+1}^\Delta \xrightarrow{\pi_2} Z_{p+1,q+1}^\Delta, \quad (3)$$

where π_1 and π_2 are homotopy equivalences and the map ζ is induced by the stabilization map.

The first $[(p+1)/2]$ terms of the increasing filtration on $\tilde{Z}_{p,q}^\Delta$ coincide with the corresponding terms for the degenerate resolution $Z_{p,q}^\Delta$. Hence, the E^1 -terms of the spectral sequences associated with the

four resolutions above all coincide in the strip $r \leq [(p+1)/2]$. It is also obvious that the maps π_1 and π_2 in (3) induce the identity map in this strip. The fact that for $r \leq [(p+1)/2]$ the homomorphism of the E^1 -terms induced by ζ is identity as well, follows from the explicit description of the filtration on $Z_{p,q}^A$ obtained above; essentially this is just the Thom isomorphism.

Due to the action of the differentials which go from the stable to the unstable part of the spectral sequence, the stable region for the terms E^2, E^3, \dots is smaller than that for the term E^1 . A straightforward check shows that the set

$$r \leq \left\lfloor \frac{p+1}{2} \right\rfloor,$$

$$2(n-m+1)r \leq s \leq (2n-2m+1) \left(\left\lfloor \frac{p+1}{2} \right\rfloor + 1 \right) + r$$

is in the stable region for the term E^∞ .

For all $i < (2n-2m+1)([(p+1)/2] + 1)$ the graded group associated to $H_i(\mathbf{Rat}_f(p, q))$ is a direct sum of stable entries of the term E^∞ of the spectral sequence. Hence, we obtain

Proposition 8. *The map*

$$H_i(\mathbf{Rat}_f(p, q)) \rightarrow H_i(\mathbf{Rat}_f(p+1, q+1))$$

induced by the stabilization map is an isomorphism for all

$$i < (2n-2m+1) \left(\left\lfloor \frac{p+1}{2} \right\rfloor + 1 \right).$$

Together with Proposition 3 this implies Theorem 1.

5. Spaces of free maps

The proof of Theorem 2 consists in applying Theorem 1 fibrewise to the map

$$R : \mathbf{Rat}_{(d)}^{m,n} \rightarrow \mathbf{Rat}_{(d)}^{m-1,n}$$

induced by restriction to a hyperplane. In what follows we denote by D the stabilization dimension $(2n-2m+1)((d+1)/2 + 1)$.

The restriction to a hyperplane $r : \text{Map}_d(m, n) \rightarrow \text{Map}_d(m-1, n)$ is a Serre fibration with the fibre $\Omega^{2m} \mathbf{CP}^n$. Denote by $t : T \rightarrow \mathbf{Rat}_{(d)}^{m-1,n}$ the pullback of r to $\mathbf{Rat}_{(d)}^{m-1,n}$. A comparison of the spectral sequences for both fibrations shows that the inclusion $T \rightarrow \text{Map}_d(m, n)$ induces isomorphisms in homology in dimensions smaller than D . Therefore, it is sufficient to prove that the inclusion $\mathbf{Rat}_{(d)}^{m,n} \rightarrow T$ is an isomorphism on homology in these dimensions.

According to Theorem 1, the fibres of the map R have the same homology as $\Omega^{2m} \mathbf{CP}^n$ in dimensions smaller than D . Moreover, we have the following:

Lemma 9. *There exists a finite filtration $F_0 \subset F_1 \subset \dots \subset F_q = \mathbf{Rat}_{(d)}^{m-1,n}$ such that (a) the map R is a fibration over each connected component of $F_{i+1} \setminus F_i$ and (b) each F_i and $R^{-1}F_i$ are absolute neighbourhood retracts in F_{i+1} and $R^{-1}F_{i+1}$, respectively.*

Proof. There exist smooth complex algebraic varieties M and N and a map $R' : M \rightarrow N$ such that $\mathbf{Rat}_{(d)}^{m,n}$ and $\mathbf{Rat}_{(d)}^{m-1,n}$ are complements to closed subvarieties of M and N , respectively, and such that R' restricts to R on $\mathbf{Rat}_{(d)}^{m,n}$.

Indeed, the space $\mathbf{Rat}_{(d)}^{m,n}$ is a complement to a discriminant in a complex projective space; the homogeneous coordinates in this projective space are the coefficients of the polynomials representing a point in $\mathbf{Rat}_{(d)}^{m,n}$. From this point of view, the map R is just a linear projection. Blowing up the indeterminacy locus of this projection one gets the variety M ; the variety N is just a complex projective space.

There exist stratifications on M and N , with $\mathbf{Rat}_{(d)}^{m,n}$ being the maximal stratum of M , which turn $R' : M \rightarrow N$ into a stratified map, [2, Section I.1.7]. Take F_i to be the filtration on $\mathbf{Rat}_{(d)}^{m-1,n}$ produced by intersecting $\mathbf{Rat}_{(d)}^{m-1,n}$ with the above stratification of N . It follows from Thom's first isotopy lemma (see [2, I.1.5]) that R is a fibration over each component of $F_{i+1} \setminus F_i$. Part (b) of the Lemma follows from the fact that every stratified set has a triangulation compatible with the stratification.

The inclusion

$$R^{-1}F_{i+1} \setminus R^{-1}F_i \hookrightarrow t^{-1}F_{i+1} \setminus t^{-1}F_i$$

is a morphism of fibrations over the same base, hence, by Theorem 1, induces isomorphisms of homology groups in dimensions less than D . Consider the Mayer–Vietoris sequence relating the homology of $R^{-1}F_{i+1}/R^{-1}F_i$ to the homology of $R^{-1}F_{i+1} \setminus R^{-1}F_i$, $R^{-1}V/R^{-1}F_i$ and $R^{-1}\partial V$ where V is a contractible neighbourhood of F_i in F_{i+1} . (Notice that $R^{-1}V/R^{-1}F_i$ is contractible and that $R : R^{-1}\partial V \rightarrow \partial V$ is a fibration.) Comparing it to the analogous Mayer–Vietoris sequence with R replaced by t , one verifies that the homomorphisms

$$H_k(R^{-1}F_{i+1}/R^{-1}F_i) \rightarrow H_k(t^{-1}F_{i+1}/t^{-1}F_i)$$

are isomorphisms for $k < D$. Now, to finish the proof of Theorem 2, compare the spectral sequences associated to the filtrations $t^{-1}F_i$ on T , and $R^{-1}F_i$ on $\mathbf{Rat}_{(d)}^{m,n}$. \square

6. Final remarks

The estimate for the stabilization dimension in Theorem 1 in the case $m = 1$ is weaker than the estimate given by Segal's theorem. It is probable that Lemma 7 on which our estimate is based, can be replaced by a stronger statement. However, the methods of the present paper are sufficient to recover Segal's theorem, at least for $n > 1$. In the case $m = 1$, $q = 0$ the simplicial resolution $Z_{p,0}^d$ is, in fact, non-degenerate, as a polynomial of degree p in one variable can have at most p roots. Thus, the explicit description of Z_r/Z_{r-1} as a Thom space of a bundle over a configuration space is valid not only for $r \leq [(p+1)/2]$, but for all $r \leq p$. This leads to a better estimate for the stabilization dimension.

There is no doubt that Theorem 1 can be also strengthened by replacing “homology” with “homotopy” in the case $m = n$. This would require an argument similar to that used by Segal in [11]. Such an argument, however, is beyond the scope of the present paper.

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